Second homology groups and universal coverings of Steinberg Leibniz algebras of small characteristic[1]

(Appeared in *Comm. Algebra* **37** (2009), no. 2, 548-566.)

Qifen Jiang[1)], **Ran Shen**[1)] **and Yucai Su**[2)]

[1)] *Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China*

E-mail: qfjiang@sjtu.edu.cn

[2)] *Department of Mathematics, University of Science and Technology of China*

Hefei 230026, China

E-mail: ycsu@ustc.edu.cn

Abstract

It is known that the second Leibniz homology group $HL_2(\mathfrak{stl}_n(R))$ of the Steinberg Leibniz algebra $\mathfrak{stl}_n(R)$ is trivial for $n \geq 5$. In this paper, we determine $HL_2(\mathfrak{stl}_n(R))$ explicitly (which are shown to be not necessarily trivial) for $n = 3, 4$ without any assumption on the base ring.

§1 Introduction

The concept of Leibniz algebras was introduced by Loday [Lo] in the study of Leibniz homology as a noncommutative analog of Lie algebras homology. A Leibniz algebra L is a vector space equipped with a K-bilinear map $[\ ,\ ]\colon L \times L \longrightarrow L$ satisfying the Leibniz identity $[x, [y, z]] = [[x, y], z] - [[x, z], y]$ for all $x, y, z \in L$, where K is a unital commutative ring. Clearly, a Lie algebra is a Leibniz algebra. For any Leibniz algebra L there is an associated Lie algebra $L_{Lie} = L/\langle [x, x] \rangle$, where $\langle [x, x] \rangle$ is the two-sided ideal generated by all $[x, x], x \in L$. To study the second Leibniz homology group of Lie algebra $sl_n(R)$ and Steinberg Lie algebra $\mathfrak{st}_n(R)$, Loday and Pirashvili [LP] introduced also the noncommutative Steinberg Leibniz algebra $\mathfrak{stl}_n(R)$, where R is an associative algebra over a commutative ring K. In [L], Steinberg Leibniz algebra and its superalgebra were dicussed. Steinberg Lie algebras and their universal central extensions have been studied by many authors (e.g., [B1, KL, Ka, G1, G2, GS]). In most situations, the Steinberg Lie algebra $\mathfrak{st}_n(R)$ is the universal central extension of the Lie algebra $sl_n(R)$ whose kernel is isomorphic to the first cyclic homology group $HC_1(R)$ of the associative algebra R and the second Lie algebra homology group $H_2(\mathfrak{st}_n(R)) = 0$. In [Bl] and [KL], it was proved that $H_2(\mathfrak{st}_n(R)) = 0$ for $n \geq 5$. In [KL], it was mentioned without proof that $H_2(\mathfrak{st}_n(R)) = 0$ for $n = 3, 4$ if $\frac{1}{2} \in$ the base ring K. This was proved in [G1] for $n = 3$ if $\frac{1}{6} \in K$ and for $n = 4$ if $\frac{1}{2} \in K$. Gao and Shang [GS] generalize

[1]This work is supported by NSF grants 10571120, 10471096 of China and "One Hundred Talents Program" from University of Science and Technology of China.

2000 Mathematics Subject Classification: 17A32, 17B55, 17B60.

the result without any assumption on K. Since $\mathfrak{stl}_n(R)_{Lie} = \mathfrak{st}_n(R)$, it is natural to consider the Second homology group $HL_2(\mathfrak{stl}_n(R))$ of the Steinberg Leibniz algebra $\mathfrak{stl}_n(R)$ in the category of Leibniz algebras. In [LP], Loday and Pirashvili proved that $HL_2(\mathfrak{stl}_n(R)) = 0$ for $n \geq 5$.

Motivated by [GS], in this paper, we will determine $HL_2(\mathfrak{stl}_n(R))$ explicitly for $n = 3, 4$ (which are not necessarily trivial). It is equivalent to work on the Steinberg Leibniz algebras $\mathfrak{stl}_n(R)$ of small characteristic for small n. This completes the determination of the universal central extension of the Leibniz algebras $\mathfrak{stl}_n(R)$ and $sl_n(R)$ as well. We would like to remark that since skew-symmetry does not hold for Leibniz algebras, some different approach to solve the problem seems to be necessary. This is also one of our motivation to present this paper. The main result in this paper is the following theorem (cf. Theorem 3.5 and 4.4), where the result for the first case was obtained in [LP].

Theorem 1.1 *The second homology group of Steiberg Leibniz algebra* $\mathfrak{stl}_n(R)$ (*cf. Definition* 2.4) *is*

$$HL_2(\mathfrak{stl}_n(R)) = \begin{cases} 0 & \textit{if} \quad n \geq 5, \\ R_2^6 & \textit{if} \quad n = 4, \\ R_3^6 & \textit{if} \quad n = 3, \end{cases}$$

where R_2^6, R_3^6 *are defined in Definition* 3.2 *and Definition* 4.1.

The paper is organized as follows. In Section 2, we review some basic definitions and results on Steinberg Leibniz algebras $\mathfrak{stl}_n(R)$. Section 3 will discuss the $n = 4$ case. Section 4 will handle the $n = 3$ case.

§2 Preliminary Let K be a unital commutative ring.

Definition 2.1 A *Leibniz algebra* L is a vector space equipped with a K-bilinear map $[\cdot,\cdot] : L \times L \to L$, satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad \text{for all} \quad x, y, z \in L. \tag{2.1}$$

Clearly, a Lie algebra is a Leibniz algebra. For any Leibniz algebra L there is an associated Lie algebra $L_{Lie} = L/\langle [x, x] \rangle$, where $\langle [x, x] \rangle$ is the two-sided ideal generated by all $[x, x], x \in L$.

Definition 2.2 Let L be a Leibniz algebra over K, defined the boundary map $\delta_n : L^{\otimes n} \to L^{\otimes n-1}$ by

$$\delta_n(x_1 \otimes x_2 \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} (-1)^{j+1} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \hat{x_j} \otimes \cdots x_n,$$

where $\hat{x_j}$ means that the element x_j is omitted. The complex $(L^{\otimes n}, \delta)$ (where $L^0 = K$ and $\delta_1 = 0$) gives the Leibniz homology $HL_\star(L)$ of the Leibniz algebra L and $HL_n(L) = \mathrm{Ker}\, \delta_n / \mathrm{Im}\, \delta_{n+1}$ is called the n-th homology group of L.

Let L be a Leibniz algebra over K, the *center* of L is defined to be

$$\{z \in L \mid [z, L] = [L, z] = 0\}.$$

A Leibniz algebra L is called *perfect* if $[L, L] = L$. A *central extension* of L is a pair $(\hat{L}, \pi)$ where $\hat{L}$ is a Leibniz algebra, and $\pi : \hat{L} \to L$ is a surjective homomorphism such that $\mathrm{Ker}\, \pi$ lies in the center of $\hat{L}$ and the exact sequence

$$0 \to \mathrm{Ker}\, \pi \to \hat{L} \to L \to 0,$$

splits as K-modules. The pair $(\hat{L}, \pi)$ is a *universal central extension* of L if for every central extension $(\tilde{L}, \tau)$ of L there is a unique homomorphism $\psi : \hat{L} \to \tilde{L}$ such that $\tau \circ \psi = \pi$.

The following result can be found in [LP].

Proposition 2.3 *The universal central extension of a Leibniz algebra L exists if and only if L is perfect. The universal central extension is unique up to isomorphism. If $(\hat{L}, \pi)$ is the universal central extension of L, then $HL_2(L) \cong \mathrm{Ker}\, \pi$.*

Let R be a unital associative K-algebra. We always assume that R has a K-basis $\{r_\lambda \mid \lambda \in \Lambda\}$(where Λ is an index set), which contains the unit 1 of R. We denote by $gl_n(R)$ (where $n \geq 3$) the Leibniz algebra consisting of all $n \times n$ matrices with coefficients in R (which is in fact a Lie algebra), whose bracket is as follows.

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(ab) - \delta_{il} E_{kj}(ba),$$

for $a, b \in R, 1 \leq i, j, k, l \leq n$, where $E_{ij}(x)$ is the matrix with only non-zero element x in position (i, j).

The subalgebra $sl_n(R) := [gl_n(R), gl_n(R)]$ of $gl_n(R)$ is generated by the elements $E_{ij}(a)$, $a \in R$, $1 \leq i \neq j \leq n$, satisfying

$$[E_{ij}(a), E_{jk}(b)] = E_{ik}(ab), \tag{2.2}$$

$$[E_{ij}(a), E_{ki}(b)] = -E_{kj}(ba), \tag{2.3}$$

for i, j, k distinct and

$$[E_{ij}(a), E_{kl}(b)] = 0 \quad \text{for} \quad j \neq k, \ i \neq l. \tag{2.4}$$

Definition 2.4 [LP] For $n \geq 3$, the *Steinberg Leibniz algebra* $\mathfrak{stl}_n(R)$ is a Leibniz algebra over K defined by generators $X_{ij}(a)$, $a \in R$, $1 \leq i \neq j \leq n$, subject to the relations

$$X_{ij}(k_1 a + k_2 b) = k_1 X_{ij}(a) + k_2 X_{ij}(b) \quad \text{for} \ \ a, b \in R, \ k_1, k_2 \in K, \tag{2.5}$$

$$[X_{ij}(a), X_{jk}(b)] = X_{ik}(ab), \tag{2.6}$$

$$[X_{ij}(a), X_{ki}(b)] = -X_{kj}(ba) \quad \text{for distinct} \quad i, j, k, \tag{2.7}$$

$$[X_{ij}(a), X_{kl}(b)] = 0 \ \ \text{for} \ \ j \neq k, i \neq l, \tag{2.8}$$

where $a, b \in R$, $1 \leq i, \ j, \ k, \ l \leq n$.

One observes that both Leibniz algebras $sl_n(R)$ and $\mathfrak{stl}_n(R)$ are perfect since $1 \in R$. Define the Leibniz algebra homomorphism $\phi : \mathfrak{stl}_n(R) \to sl_n(R)$ by $\phi(X_{ij}(a)) = E_{ij}(a)$. Obviously, ϕ is an epimorphism.

The following result can be found in [LP, KL].

Theorem 2.5 *For $n \geq 3$ the kernel of ϕ is central in $\mathfrak{stl}_n(R)$ and is isomorphic to $HH_1(R)$. Moreover if $n \geq 5$ then*

$$0 \to HH_1(R) \to \mathfrak{stl}_n(R) \to sl_n(R) \to 0$$

is the universal central extension of $sl_n(R)$ (in the category of Leibniz algebras).

Here and below, $HH_1(R)$ denotes the *Hochschild homology group* of R with coefficients in R. This result means $HL_2(\mathfrak{stl}_n(R)) = 0$ for $n \geq 5$ and the universal central extension of $sl_n(R)$ is also the universal central extension of $\mathfrak{stl}_n(R)$, denoted by $\widehat{\mathfrak{stl}}_n(R)$. Our purpose is to determine $\widehat{\mathfrak{stl}}_n(R)$ for any ring K and $n \geq 3$. The following proposition can be similarly proved as in [AF] for the Steinberg unitary Lie algebra case(Jacobi identity of Lie algebra replaced by Leibniz identity).

Lemma 2.6 *Let $H := \sum_{1 \leq i \neq j \leq n} [X_{ij}(R), X_{ji}(R)]$. Then H is a subalgebra of $\mathfrak{stl}_n(R)$ containing the center of $\mathfrak{stl}_n(R)$ such that $[H, X_{ij}(R)] \subseteq X_{ij}(R)$. Moreover,*

$$\mathfrak{stl}_n(R) = H \oplus \sum_{1 \leq i \neq j \leq n} X_{ij}(R). \tag{2.9}$$

For a fixed K-basis $\{r_\lambda \,|\, \lambda \in \Lambda\}$ of R, clearly

$$\Gamma := \{X_{ij}(r_\lambda) \,|\, \lambda \in \Lambda, \, 1 \leq i \neq j \leq n\}, \tag{2.10}$$

is a K-basis of $\mathfrak{stl}_n(R)$.

We will see (cf. Lemma 2.8) that the subalgebra H has a more refined structure. Setting

$$T_{ij}(a, b) = [X_{ij}(a), X_{ji}(b)], \tag{2.11}$$

$$t(a, b) = T_{1j}(a, b) - T_{1j}(ba, 1), \tag{2.12}$$

for $a, b \in R, 1 \leq i \neq j \leq n$. Note that $T_{ij}(a, b)$ is K-bilinear, and so is $t(a, b)$. Further we have the following

Lemma 2.7 *For $a, b, c \in R$ and distinct i, j, k, we have*

(1) $T_{ij}(a, bc) = T_{ik}(ab, c) + T_{kj}(ca, b)$,

(2) $T_{kj}(c, 1) + T_{jk}(c, 1) = 0$,

(3) $t(a, b)$ *does not depend on the choice of* j.

Proof. Using Leibniz identity and the relations of generators of $\mathfrak{stl}_n(R)$, one has

$$\begin{aligned} T_{ij}(a, bc) &= [X_{ij}(a), X_{ji}(bc)] = [X_{ij}(a), [X_{jk}(b), X_{ki}(c)]] \\ &= [[X_{ij}(a), X_{jk}(b)], X_{ki}(c)] - [[X_{ij}(a), X_{ki}(c)], X_{jk}(b)] \\ &= [X_{ik}(ab), X_{ki}(c)] - [-X_{kj}(ca), X_{jk}(b)] \\ &= T_{ik}(ab, c) + T_{kj}(ca, b). \end{aligned}$$

This show that (1) holds. By the (1), we have

$$T_{ij}(ab, c) = T_{ik}(a, bc) - T_{jk}(ca, b). \tag{2.13}$$

Taking $b = 1$ in (1) and (2.13), we have $T_{ij}(a, c) = T_{ik}(a, c) + T_{kj}(ca, 1)$ and $T_{ij}(a, c) = T_{ik}(a, c) - T_{jk}(ca, 1)$. Combining the two identities, we prove (2). Taking $k \notin \{1, j\}$ and using (1), we have

$$\begin{aligned} t(a, b) &= T_{1j}(a, b) - T_{1j}(ba, 1) \\ &= T_{1k}(a, b) + T_{kj}(ba, 1) - T_{1j}(ba, 1) \\ &= T_{1k}(a, b) - (T_{1j}(ba, 1) - T_{kj}(ba, 1)) \\ &= T_{1k}(a, b) - T_{1k}(ba, 1), \end{aligned}$$

which proves (3). $\square$

Lemma 2.8 *Every $x \in H$ can be written as the following form*

$$x = \sum_i t(a_i, b_i) + \sum_{2 \le j \le n} T_{1j}(c_j, 1),$$

where $a_i, b_i, c_j \in R$.

Proof. Consider $T_{ij}(a, b)$ (since H is generated by $T_{ij}(a, b), i \neq j$). By Lemma 2.7, we have the following: Suppose $i = 1$. Take $k \notin \{1, j\}$, then

$$\begin{aligned} T_{1j}(a, b) &= T_{1k}(a, b) + T_{kj}(ba, 1) \\ &= T_{1k}(a, b) + T_{k1}(ba, 1) + T_{1j}(ba, 1) \\ &= T_{1k}(a, b) - T_{1k}(ba, 1) + T_{1j}(ba, 1)) \\ &= t(a, b) + T_{1j}(ba, 1). \end{aligned}$$

Suppose $i, j \neq 1$. Then

$$\begin{aligned}
T_{ij}(a,b) &= T_{i1}(ab,1) + T_{1j}(a,b) \\
&= -T_{1i}(ab,1) + t(a,b) + T_{1j}(ba,1) \\
&= t(a,b) - (T_{1i}(ab,1) + T_{1j}(ba,1)).
\end{aligned}$$

Suppose $i \neq 1, j = 1$. Take $k \notin \{1, i\}$, then

$$\begin{aligned}
T_{i1}(a,b) &= T_{ik}(a,b) + T_{k1}(ba,1) \\
&= T_{i1}(ab,1) + T_{1k}(a,b) - T_{1k}(ba,1) \\
&= t(a,b) - T_{1i}(ab,1).
\end{aligned}$$

This prove the lemma. □

For later use, we need the following results which are easy to check directly by Leibniz identity.

Lemma 2.9 *For distinct i, j, k, l and for $a, b, c \in R$, we have*

(1) $[T_{ij}(a,b), X_{kl}(c)] = [X_{kl}(c), T_{ij}(a,b)] = 0$,

(2) $[T_{ij}(a,b), X_{ik}(c)] = X_{ik}(abc) = -[X_{ik}(c), T_{ij}(a,b)]$,

(3) $[T_{ij}(a,b), X_{ki}(c)] = -X_{ki}(cab) = -[X_{ki}(c), T_{ij}(a,b)]$,

(4) $[T_{ij}(a,b), X_{kj}(c)] = X_{kj}(cba) = -[X_{kj}(c), T_{ij}(a,b)]$,

(5) $[T_{ij}(a,b), X_{ij}(c)] = X_{ij}(abc + cba) = -[X_{ij}(c), T_{ij}(a,b)]$,

(6) $[T_{ij}(a,b), X_{jk}(c)] = -X_{jk}(bac) = -[X_{jk}(c), T_{ij}(a,b)]$,

(7) $[t(a,b), X_{1i}(c)] = X_{1i}((ab - ba)c) = -[X_{1i}(c), t(a,b)]$,

(8) $[t(a,b), X_{i1}(c)] = -X_{i1}(c(ab - ba)) = -[X_{i1}(c), t(a,b)]$,

(9) $[t(a,b), X_{jk}(c)] = 0 = -[X_{jk}(c), t(a,b)]$ *for* $j, k \geq 2$.

§3 Universal central extension of $\mathfrak{stl}_4(R)$

In this section, we determine the universal central extension $\widehat{\mathfrak{stl}}_4(R)$ of $\mathfrak{stl}_4(R)$ and compute $HL_2(\mathfrak{stl}_4(R))$ without any assumption on R.

For any positive integer m, let $\mathcal{I}_m$ be the ideal of R generated by the elements ma and $ab - ba$, for $a, b \in R$. One immediately has

Lemma 3.1 (*cf.* [GS]) $\mathcal{I}_m = mR + R[R,R]$ *and* $[R,R]R = [R,R]R$.

Let $R_m := R/\mathcal{I}_m$ be the quotient algebra over K which is commutative. Write $\bar{a} = a + \mathcal{I}_m$ for $a \in R$. Note that if $m = 2$ then $\overline{a} = -\overline{a}$ in R_m.

Definition 3.2 We define $\mathcal{W} = R_2^6$ to be the direct sum of six copies of R_2. For $1 \le m \le 6$, we let $\epsilon_m(\overline{a}) = (0, \cdots, \overline{a}, \cdots, 0)$ be the element of $\mathcal{W}$ such that the m-th coordinate is $\overline{a}$ and zero otherwise.

Let S_4 be the symmetric group of $\{1, 2, 3, 4\}$. Let

$$P = \{(i, j, k, l) \,|\, \{i, j, k, l\} = \{1, 2, 3, 4\}\},$$

be the set of all the quadruples with the distinct components. Then S_4 has a natural transitive action on P given by

$$\sigma((i, j, k, l)) = (\sigma(i), \sigma(j), \sigma(k), \sigma(l)) \text{ for any } \sigma \in S_4.$$

Clearly

$$G = \{(1), (13), (24), (13)(24)\},$$

is a subgroup of S_4 with $[S_4 : G] = 6$. Then S_4 has a partition of cosets with respect to G, denoted by

$$S_4 = \bigsqcup_{m=1}^{6} \sigma_m G.$$

We obtain a partition of P by

$$P = \bigsqcup_{m=1}^{6} P_m, \quad \text{where } P_m = (\sigma_m G)((1, 2, 3, 4)).$$

Define the index map $\theta : P \to \{1, 2, 3, 4, 5, 6\}$ by

$$\theta\left((i, j, k, l)\right) = m \;\text{ if } (i, j, k, l) \in P_m, \quad \text{for } 1 \le m \le 6.$$

We fix $P_1 = G((1, 2, 3, 4))$, then we have $(1, 2, 3, 4) \in P_1$ and $\theta((1, 2, 3, 4)) = 1$.

Using the decomposition (2.9) of $\mathfrak{stl}_4(R)$, we take Γ as in (2.10) with $n = 4$. Define $\psi : \Gamma \times \Gamma \to \mathcal{W}$ by

$$\psi(X_{ij}(r), X_{kl}(s)) = \epsilon_{\theta((i,j,k,l))}(\overline{rs}) \in \mathcal{W} \text{ for } r, s \in \{r_\lambda \,|\, \lambda \in \Lambda\} \text{ and distinct } i, j, k, l,$$

and $\psi = 0$ otherwise. Then we obtain the K-bilinear map $\psi : \mathfrak{stl}_4(R) \times \mathfrak{stl}_4(R) \to \mathcal{W}$ by linearity. We now have

Lemma 3.3 *The bilinear map ψ is a Leibniz 2-cocycle.*

Proof. It suffices to prove

$$J(x, y, z) := \psi(x, [y, z]) + \psi([x, z], y) - \psi([x, y], z) = 0, \tag{3.1}$$

for any $x, y, z \in \mathfrak{stl}_4(R)$. According to (2.9) and Lemma 2.8, the Steinberg Leibniz algebra $\mathfrak{stl}_4(R)$ has the decomposition :

$$\mathfrak{stl}_4(R) = t(R, R) \oplus T_{12}(R, 1) \oplus T_{13}(R, 1) \oplus T_{14}(R, 1) \oplus \bigoplus_{1 \le i \ne j \le n} X_{ij}(R), \tag{3.2}$$

where $t(R, R)$ is the K-linear span of the elements $t(a, b)$. Clearly, the number of elements of x, y, z belonging to the subalgebra H such that $\psi([x, y], z) \ne 0$ is at most one. We consider the following possibilities:

Case 1: Suppose there exists exactly one of $\{x, y, z\}$ belonging to H. Say, $x = X_{12}(a), y = X_{34}(b)$ and $z \in H$, where $a, b \in R$ (we omit the other subcases since they are very similar, although not identical). We can assume that either $z = t(c, d)$, where $c, d \in R$, or $z = T_{1j}(c, 1)$, where $2 \le j \le 4$ and $c \in R$.

If $z = t(a, b)$, then according to the Leibniz identity and Lemma 2.9, we have

$$\begin{aligned} J(x, y, z)) &= \psi([X_{12}(a), t(c, d)], X_{34}(b)) \\ &= \psi(-X_{12}((cd - dc)a), X_{34}(b)) \\ &= -\epsilon_1(\overline{(cd - dc)ab}) = 0. \end{aligned}$$

If $z = T_{12}(c, 1)$, then

$$\begin{aligned} J(x, y, z)) &= \psi([X_{12}(a), T_{12}(1, c)], X_{3,4}(b)) \\ &= \psi(-X_{12}(ca + ac), X_{34}(b)) \\ &= -\epsilon_1(\overline{(ca + ac)b}) = 0. \end{aligned}$$

If $z = T_{13}(c, 1)$, then

$$\begin{aligned} J(x, y, z)) &= \psi(X_{12}(a), [X_{34}(b), T_{13}(c, 1)]) + \psi([X_{12}(a), T_{13}(c, 1)], X_{34}(b)) \\ &= \psi(X_{12}(a), X_{34}(cb)) + \psi(-X_{12}(ca), X_{34}(b)) \\ &= \epsilon_1(\overline{acb} - \overline{cab}) = \epsilon_1(\overline{c(ab - ba)}) = 0. \end{aligned}$$

If $z = T_{14}(c, 1)$, then

$$\begin{aligned} J(x, y, z) &= \psi(X_{12}(a), [X_{34}(b), T_{14}(c, 1)]) + \psi([X_{12}(a), T_{14}(c, 1)], X_{34}(b)) \\ &= \psi(X_{12}(a), -X_{34}(bc)) + \psi(-X_{12}(ca), X_{34}(b)) \\ &= -\epsilon_1(\overline{abc} + \overline{cab}) = -\epsilon_1(\overline{abc - abc}) = 0. \end{aligned}$$

Case 2: Suppose none of $\{x, y, z\}$ belongs to H. The nonzero terms of $J(x, y, z)$ must be $\psi([X_{ik}(a), X_{kj}(b)], X_{kl}(c))$ or $\psi([X_{il}(a), X_{lj}(b)], X_{kl}(c))$ for distinct i, j, k, l and $a, b, c \in R$.

In case $x = X_{ik}(a)$, $y = X_{kj}(b)$ and $z = X_{kl}(c)$, we have

$$\begin{aligned} J(x,y,z)) &= \psi([X_{ik}(a), X_{kl}(c)], X_{kj}(b)) - \psi([X_{ik}(a), X_{kj}(b)], X_{kl}(c)) \\ &= \psi(X_{il}(ac), X_{kj}(b)) - \psi(X_{ij}(ab), X_{kl}(c)) \\ &= \epsilon_{\theta((i,l,k,j))}(\overline{acb}) - \epsilon_{\theta((i,j,k,l))}(\overline{abc}) \\ &= \epsilon_{\theta((i,j,k,l))}(\overline{a(cb-bc)}) = 0. \end{aligned}$$

In case $x = X_{il}(a)$,$y = X_{lj}(b)$ and $z = X_{kl}(c)$, we have

$$\begin{aligned} J(x,y,z)) &= \psi(X_{il}(a), [X_{lj}(b), X_{kl}(c)]) - \psi([X_{il}(a), X_{lj}(b)], X_{kl}(c)) \\ &= \psi(X_{il}(a), -X_{kj}(cb)) - \psi(X_{ij}(ab), X_{kl}(c)) \\ &= -\epsilon_{\theta((i,l,k,j))}(\overline{acb}) - \epsilon_{\theta((i,j,k,l))}(\overline{abc}) \\ &= -\epsilon_{\theta((i,j,k,l))}(\overline{a(bc+cb)}) = 0, \end{aligned}$$

where the fourth equality follows from the fact that (i,j,k,l) and (i,l,k,j) are in the same partition of P, i.e. $\theta((i,j,k,l)) = \theta((k,l,i,j))$. This is because that if $(ijkl) = \sigma((1234))$ (where $\sigma \in S_4$), then $(ilkj) = (\sigma \circ (24))((1234))$. The proof is completed. □

We therefore obtain a central extension of Leibniz algebra $\mathfrak{stl}_4(R)$:

$$0 \to \mathcal{W} \to \widehat{\mathfrak{stl}}_4(R) \xrightarrow{\pi} \mathfrak{stl}_4(R) \to 0, \tag{3.3}$$

i.e.

$$\widehat{\mathfrak{stl}}_4(R) = \mathcal{W} \oplus \mathfrak{stl}_4(R), \tag{3.4}$$

with bracket

$$[(c,x),(c',y)] = (\psi(x,y),[x,y]),$$

for all $x, y \in \mathfrak{stl}_4(R)$ and $c, c' \in \mathcal{W}$, where $\pi : \mathcal{W} \oplus \mathfrak{stl}_4(R) \to \mathfrak{stl}_4(R)$ is the second coordinate projection map. Namely, $(\widehat{\mathfrak{stl}}_4(R), \pi)$ is a central extension of $\mathfrak{stl}_4(R)$. We will show that $(\widehat{\mathfrak{stl}}_4(R), \pi)$ is the universal central extension of $\mathfrak{stl}_4(R)$. To do this, we define a Leibniz algebra $\mathfrak{stl}_4(R)^\sharp$ to be the Leibniz algebra generated by the symbols $X^\sharp_{ij}(a)$, $a \in R$ and the K-linear space $\mathcal{W}$, satisfying the following relations:

$$X_{ij}{}^\sharp(k_1a + k_2b) = k_1 X_{ij}{}^\sharp(a) + k_2 X_{ij}{}^\sharp(b) \text{ for } a, b \in R,\ k_1, k_2 \in K, \tag{3.5}$$

$$[X^\sharp_{ij}(a), X^\sharp_{jk}(b)] = -[X^\sharp_{jk}(b), X^\sharp_{ij}(a)] = X^\sharp_{ik}(ab) \text{ for distinct } i,j,k, \tag{3.6}$$

$$[X^\sharp_{ij}(a), \mathcal{W}] = 0 = [\mathcal{W}, X^\sharp_{ij}(a)] \text{ for distinct } i,j, \tag{3.7}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ij}(b)] = 0 \text{ for distinct } i,j, \tag{3.8}$$

$$[X^\sharp_{ij}(a), X^\sharp_{ik}(b)] = 0 \text{ for distinct } i,j,k, \tag{3.9}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kj}(b)] = 0 \text{ for distinct } i,j,k, \tag{3.10}$$

$$[X^\sharp_{ij}(a), X^\sharp_{kl}(b)] = \epsilon_{\theta((i,j,k,l))}(\overline{ab}) \text{ for distinct } j,k,i,l, \tag{3.11}$$

where $a, b \in R$, $1 \leq i, j, k, l \leq 4$. Since $1 \in R$, we see $\mathfrak{stl}_4(R)^\sharp$ is perfect. Clearly, there is a unique Leibniz algebra homomorphism $\rho : \mathfrak{stl}_4(R)^\sharp \to \widehat{\mathfrak{stl}}_4(R)$ such that $\rho(X_{ij}^\sharp(a)) = X_{ij}(a)$ and $\rho|_{\mathcal{W}} = id$. One can easily observe that ρ is actually an isomorphism. Namely,

Lemma 3.4 *$\rho : \mathfrak{stl}_4(R)^\sharp \to \widehat{\mathfrak{stl}}_4(R)$ is a Leibniz algebra isomorphism.*

The analogue of the following theorem for Lie algebra was obtained in [GS]. However, in our case, since the skew-symmetry does not hold for Leibniz algebras, we need to find some different approach to solve the problem.

Theorem 3.5 *$(\widehat{\mathfrak{stl}}_4(R), \pi)$ is the universal central extension of $\mathfrak{stl}_4(R)$ and hence*

$$HL_2(\mathfrak{stl}_4(R)) \cong \mathcal{W}.$$

Proof. Suppose

$$0 \to \mathcal{V} \to \widetilde{\mathfrak{stl}}_4(R) \xrightarrow{\tau} \mathfrak{stl}_4(R) \to 0,$$

is a central extension of $\mathfrak{stl}_4(R)$. We must show that there exists a Leibniz algebra homomorphism $\eta : \widehat{\mathfrak{stl}}_4(R) \to \widetilde{\mathfrak{stl}}_4(R)$ such that $\tau \circ \eta = \pi$. By Lemma 3.4, it suffices to show that there exists a Leibniz algebra homomorphism $\xi : \mathfrak{stl}_4(R)^\sharp \to \widetilde{\mathfrak{stl}}_4(R)$ such that $\tau \circ \xi = \pi \circ \rho$.

Using the K-basis $\{r_\lambda \mid \lambda \in \Lambda\}$ of R, we choose a preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ under τ for $1 \leq i \neq j \leq 4$ and $a \in \{r_\lambda \mid \lambda \in \Lambda\}$. For distinct i, j, k, let

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab) + \mu_{ij}^k(a, b),$$

where $\mu_{ij}^k(a, b) \in \mathcal{V}$. Similar to the discussions in [GS] (replacing of the Jacobi identity by Leibniz identity (2.1)), we obtain that $\mu_{ij}^k(a, b)$ is independent of the choice of k and by re-choosing the preimage $\widetilde{X}_{ik}(a)$, we can suppose as in [GS],

$$[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab). \tag{3.12}$$

We also need to consider the bracket $[\widetilde{X}_{kj}(b), \widetilde{X}_{ik}(a)]$ for $a, b \in R$ and distinct i, j, k since there is no skew-symmetry for Leibniz algebras. Our approach is different from that in [GS]. Let

$$[\widetilde{X}_{kj}(a), \widetilde{X}_{ik}(b)] = -\widetilde{X}_{ij}(ba) + \widetilde{\mu_{ij}}(a, b),$$

where $\widetilde{\mu_{ij}}(a, b) \in \mathcal{V}$ is independent of k as in $\mu_{ij}^k(a, b)$. Take distinct i, j, k, l, then

$$\Big[[\widetilde{X}_{kl}(a), \widetilde{X}_{lj}(c)], \widetilde{X}_{ik}(b)\Big] = [\widetilde{X}_{kj}(ac), \widetilde{X}_{ik}(b)]. \tag{3.13}$$

The left-hand side of (3.13) is, by Leibniz identity (2.1),

$$\Big[\widetilde{X}_{kl}(a), [\widetilde{X}_{lj}(c), \widetilde{X}_{ik}(b)]\Big] + \Big[[\widetilde{X}_{kl}(a), \widetilde{X}_{ik}(b)], \widetilde{X}_{lj}(c)\Big] = [-\widetilde{X}_{il}(ba), \widetilde{X}_{lj}(c)] = -\widetilde{X}_{ij}(bac),$$

since $[\widetilde{X}_{lj}(c), \widetilde{X}_{ik}(b)] \in \mathcal{V}$. On the other hand, the right-hand side of (3.13) is $-\widetilde{X}_{ij}(bac) + \widetilde{\mu_{ij}}(ac, b)$. Thus $\widetilde{\mu_{ij}}(ac, b) = 0$. In particular, taking $c = 1$, we have $\widetilde{\mu_{ij}}(a, b) = 0$. Therefore

$$[\widetilde{X}_{kj}(a), \widetilde{X}_{ik}(b)] = -\widetilde{X}_{ij}(ba), \tag{3.14}$$

for $a, b \in R$ and distinct $i,\ j,\ k$. Now (3.12) and (3.14) imply (3.6).

Next for $k \neq i, k \neq j$, we have

$$\begin{aligned}[\widetilde{X}_{ij}(a), \widetilde{X}_{ij}(b)] &= \Big[\widetilde{X}_{ij}(a), [\widetilde{X}_{ik}(b), \widetilde{X}_{kj}(1)]\Big] \\ &= \Big[[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)], \widetilde{X}_{kj}(1)\Big] - \Big[[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(1)], \widetilde{X}_{ik}(b)\Big] \\ &= 0 - 0 = 0, \end{aligned} \tag{3.15}$$

as both of $[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)]$ and $[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(1)]$ are in $\mathcal{V}$. Thus, relation (3.8) is obtained.

To obtain relation (3.9), take $l \notin \{i, j, k\}$, then by Leibniz identity (2.1),

$$\begin{aligned}[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] &= \Big[\widetilde{X}_{ij}(a), [\widetilde{X}_{il}(b), \widetilde{X}_{lk}(1)]\Big] \\ &= \Big[[\widetilde{X}_{ij}(a), \widetilde{X}_{il}(b)], \widetilde{X}_{lk}(1)\Big] - \Big[[\widetilde{X}_{ij}(a), \widetilde{X}_{lk}(1)], \widetilde{X}_{il}(b)\Big] \\ &= 0 - 0 = 0, \end{aligned} \tag{3.16}$$

since $[\widetilde{X}_{ij}(a), \widetilde{X}_{il}(b)], [\widetilde{X}_{ij}(a), \widetilde{X}_{lk}(1)] \in \mathcal{V}$. Similarly, we have

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(b)] = 0, \tag{3.17}$$

for distinct i, j, k, i.e., we have relation (3.10).

To verify (3.11), set $\widetilde{T}_{ij}(a, b) = [\widetilde{X}_{ij}(a), \widetilde{X}_{ji}(b)]$. The following brackets are easily checked by the Leibniz identity.

$$\begin{aligned}&[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ik}(c)] = \widetilde{X}_{ik}(abc) = -[\widetilde{X}_{ik}(c), \widetilde{T}_{ij}(a, b)], \\ &[\widetilde{T}_{ij}(a, b), \widetilde{X}_{kj}(c)] = \widetilde{X}_{kj}(cba) = -[\widetilde{X}_{kj}(c), \widetilde{T}_{ij}(a, b)], \text{and} \\ &[\widetilde{T}_{ij}(a, b), \widetilde{X}_{kl}(c)] = 0 = [\widetilde{X}_{kl}(c), \widetilde{T}_{ij}(a, b)]. \end{aligned} \tag{3.18}$$

Then we have

$$\begin{aligned}[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ij}(c)] &= \Big[\widetilde{T}_{ij}(a, b), [\widetilde{X}_{ik}(c), \widetilde{X}_{kj}(1)]\Big] \\ &= \Big[[\widetilde{T}_{ij}(a, b), \widetilde{X}_{ik}(c)], \widetilde{X}_{kj}(1)\Big] - \Big[[\widetilde{T}_{ij}(a, b), \widetilde{X}_{kj}(1)], \widetilde{X}_{ik}(c)\Big] \\ &= \widetilde{X}_{ij}(abc) - (-\widetilde{X}_{ij}(cba)) = \widetilde{X}_{ij}(abc + cba), \end{aligned} \tag{3.19}$$

for $a, b, c \in R$ and distinct i, j, k, l. Similarly

$$[\widetilde{X}_{ij}(c), \widetilde{T}_{ij}(a, b)] = -\widetilde{X}_{ij}(abc + cba).$$

Next, for distinct i, j, k, l, let

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)] = \nu_{kl}^{ij}(a,b),$$

where $\nu_{kl}^{ij}(a,b) \in \mathcal{V}$. By (3.18) and (3.19),

$$\begin{aligned}
2\nu_{kl}^{ij}(a,b) &= [\widetilde{X}_{ij}(2a), \widetilde{X}_{kl}(b)] = \Big[[\widetilde{T}_{ij}(1,1), \widetilde{X}_{ij}(a)], \widetilde{X}_{kl}(b)\Big] \\
&= \Big[\widetilde{T}_{ij}(1,1), [\widetilde{X}_{ij}(a), \widetilde{X}_{kl}(b)]\Big] + \Big[[\widetilde{T}_{ij}(1,1), \widetilde{X}_{kl}(b)], \widetilde{X}_{ij}(a)\Big] \\
&= 0 + 0 = 0.
\end{aligned}$$

So

$$\nu_{kl}^{ij}(a,b) = -\nu_{kl}^{ij}(a,b). \tag{3.20}$$

Using Leibniz identity (2.1), we have

$$\begin{aligned}
\nu_{kj}^{il}(bc,a) &= \Big[[\widetilde{X}_{ik}(b), \widetilde{X}_{kl}(c)], \widetilde{X}_{kj}(a)\Big] \\
&= \Big[\widetilde{X}_{ik}(b), [\widetilde{X}_{kl}(c), \widetilde{X}_{kj}(a)]\Big] + \Big[[\widetilde{X}_{ik}(b), \widetilde{X}_{kj}(a)], \widetilde{X}_{kl}(c)\Big] \\
&= [\widetilde{X}_{ij}(ba), \widetilde{X}_{kl}(c)] = \nu_{kl}^{ij}(ba,c).
\end{aligned}$$

Taking $c = 1$ and $b = 1$ respectively, we have $\nu_{kj}^{il}(b,a) = \nu_{kl}^{ij}(ba,1)$ and $\nu_{kj}^{il}(c,a) = \nu_{kl}^{ij}(a,c)$. It then follows that

$$\nu_{kj}^{il}(b,a) = \nu_{kl}^{ij}(a,b) = \nu_{kl}^{ij}(ba,1), \tag{3.21}$$

where $a, b \in R$ and i, j, k, l are distinct. So $\nu_{kl}^{ij}(R,R) = \nu_{kl}^{ij}(R,1)$.

Moreover, by (3.19), we get

$$\begin{aligned}
\nu_{kl}^{ij}(abc+cba,d) &= [\widetilde{X}_{ij}(abc+cba), \widetilde{X}_{kl}(d)] = \Big[[\widetilde{T}_{ij}(a,b), \widetilde{X}_{ij}(c)], \widetilde{X}_{kl}(d)\Big] \\
&= \Big[\widetilde{T}_{ij}(a,b), [\widetilde{X}_{ij}(c), \widetilde{X}_{kl}(d)]\Big] + \Big[[\widetilde{T}_{ij}(a,b), \widetilde{X}_{kl}(d)], \widetilde{X}_{ij}(c)\Big] \\
&= [\widetilde{T}_{ij}(a,b), \nu_{kl}^{ij}(c,d)] + [0, \widetilde{X}_{ij}(c)] = 0. \qquad (3.22)
\end{aligned}$$

Taking $c = d = 1$ gives $\nu_{kl}^{ij}(ab+ba,1) = 0$, i.e.

$$\nu_{kl}^{ij}(a,b) = \nu_{kl}^{ij}(ba,1) = -\nu_{kl}^{ij}(ab,1) = \nu_{kl}^{ij}(ab,1) = \nu_{kl}^{ij}(b,a). \tag{3.23}$$

Letting $c = 1$ in (3.22) and using (3.21) and (3.20), we get

$$\begin{aligned}
\nu_{kl}^{ij}(d(ab-ba),1) &= \nu_{kl}^{ij}(ab-ba,d) = \nu_{kl}^{ij}(ab,d) - \nu_{kl}^{ij}(ba,d) \\
&= \nu_{kl}^{ij}(ab,d) + \nu_{kl}^{ij}(ba,d) \\
&= \nu_{kl}^{ij}(ab+ba,d) = 0, \qquad (3.24)
\end{aligned}$$

for $a, b, c, d \in R$ and distinct i, j, k, l.

As for the other equalities of (3.11), let

$$[\widetilde{X}_{kl}(b), \widetilde{X}_{ij}(a)] = -\nu_{kl}^{ij}(a, b) + \nu',$$

where $\nu_{kl}^{ij}(a, b), \nu' \in \mathcal{V}$. By Leibniz identity (2.1) and (3.21), (3.20), we have

$$\begin{aligned}
\Big[\widetilde{X}_{kl}(b), [\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(1)]\Big] &= \Big[[\widetilde{X}_{kl}(b), \widetilde{X}_{ik}(a)], \widetilde{X}_{kj}(1)\Big] - \Big[[\widetilde{X}_{kl}(b), \widetilde{X}_{kj}(1)], \widetilde{X}_{ik}(a)\Big] \\
&= [-\widetilde{X}_{il}(ab), \widetilde{X}_{kj}(1)] - 0 \\
&= -\nu_{kj}^{il}(ab, 1) = -\nu_{kj}^{il}(b, a) = -\nu_{kl}^{ij}(a, b).
\end{aligned}$$

So $\nu' = 0$, and

$$[\widetilde{X}_{kl}(b), \widetilde{X}_{ij}(a)] = -\nu_{kl}^{ij}(a, b). \tag{3.25}$$

Note that, (3.20) and (3.24) show

$$\nu_{kl}^{ij}(\mathcal{I}_2, 1) = 0, \tag{3.26}$$

where $\mathcal{I}_2 = \text{Span}\{2a, c(ab - ba) | a, b, c \in R)\}$ (cf. Lemma 3.1). Moreover, for any $a \in R$, by (3.20), (3.21), (3.23) and (3.25), we get

$$\nu_{34}^{12}(a, 1) = \nu_{32}^{14}(a, 1) = \nu_{12}^{34}(a, 1) = \nu_{14}^{32}(a, 1).$$

It shows that the subgroup $G = \{(1), (13), (24), (13)(24)\}$ of S_4 fixes $\nu_{34}^{12}(a, 1)$. Now similar arguments as in [GS] complete the proof of the theorem. $\square$

§4 The second homology group $HL_2(\mathfrak{stl}_3(R))$ of $\mathfrak{stl}_3(R)$

In this section we compute $HL_2(\mathfrak{stl}_3(R))$. Recall from Lemma 3.1 that $\mathcal{I}_3 = 3R + R[R, R]$, and $R_3 = R/\mathcal{I}_3$ is an associative commutative K-algebra.

Definition 4.1 *Denote $\mathcal{U} = R_3^6$, and we also use $R_3^{(i)}, i = -3, -2, \cdots, 3$ to denote a copy of R_3. For $\overline{a} \in R_3$, $\overline{a}^{(i)}$ will denote the corresponding element $(0, \cdots, \overline{a}, \cdots, 0)$ in $\mathcal{U}$.*

For convenience, for $1 \leq m \neq n \leq 3$ we use the symbol:

$$\text{sign}(m, n) = \begin{cases} 1, & \text{if } m < n, \\ -1, & \text{if } m > n. \end{cases}$$

Take Γ as in (2.10) with $n = 3$. We define $\psi : \Gamma \times \Gamma \to \mathcal{U}$ by

$$\psi(X_{ij}(r), X_{ik}(s)) = \text{sign}(j, k)(\overline{rs})^{(i)}$$

$$\psi(X_{ij}(r), X_{kj}(s)) = \text{sign}(i, k)(\overline{rs})^{(-j)},$$

for $r, s \in \{r_\lambda \mid \lambda \in \Lambda\}$ and distinct i, j, k, and $\psi(x, y) = 0$ otherwise. Then ψ can be extended to a bilinear map $\mathfrak{stl}_3(R) \times \mathfrak{stl}_3(R) \to \mathcal{U}$. We have

Lemma 4.2 *The bilinear map ψ is a Leibniz 2-cocycle.*

Proof: Similar to the proof of Lemma 3.3, we show $J(x,y,z)=0$ for $x,y,z\in\mathfrak{stl}_3(R)$ (cf. (3.1)). We have the decomposition,

$$\mathfrak{stl}_3(R)=t(R,R)\oplus T_{12}(R,1)\oplus T_{13}(R,1)\oplus\bigoplus_{1\le i\ne j\le 3}X_{ij}(R). \tag{4.1}$$

As in the proof of Lemma 3.3, we can suppose at most one of $\{x,y,z\}$ is in the subalgebra H. We consider the following cases.

Case 1: Suppose $z\in H$.

We first verify two subcases $x=X_{12}(a),y=X_{13}(b)$ or $x=X_{21}(a),y=X_{31}(b)$ for $a,b\in R$. By (4.1), we may assume that either $z=t(c,d)$, where $c,d\in R$, or $z=T_{1j}(c,1)$, where $2\le j\le 3$ and $c\in R$.

In case $x=X_{12}(a),y=X_{13}(b)$, we have, according to the Leibniz identity, if $z=t(c,d)$, then

$$\begin{aligned}J(x,y,z)) &= \psi(X_{12}(a),[X_{13}(b),t(c,d)])+\psi([X_{12}(a),t(c,d)],X_{13}(b))\\ &= -\psi(X_{12}(a),X_{13}((cd-dc)b))-\psi(X_{12}((cd-dc)a),X_{13}(b))\\ &= -(\overline{a(cd-dc)b+(cd-dc)ab})^{(1)}=0.\end{aligned}$$

If $z=T_{12}(c,1)$, then

$$\begin{aligned}J(x,y,z)) &= \psi(X_{12}(a),[X_{13}(b),T_{12}(c,1)])+\psi([X_{12}(a),T_{12}(c,1)],X_{13}(b))\\ &= \psi(X_{12}(a),-X_{13}(cb))+\psi(-X_{12}(ca+ac),X_{13}(b))\\ &= -(\overline{acb+(ca+ac)b})^{1}=-(\overline{3abc})^{(1)}=0.\end{aligned}$$

If $z=T_{13}(c,1)$, then

$$\begin{aligned}J(x,y,z)) &= \psi(X_{12}(a),[X_{13}(b),T_{13}(c,1)])+\psi([X_{12}(a),T_{13}(c,1)],X_{13}(b))\\ &= \psi(X_{12}(a),-X_{13}(cb+bc))+\psi(-X_{12}(ca),X_{13}(b))\\ &= -(\overline{a(cb+bc)+cab})^{1}=-(\overline{3abc})^{(1)}=0.\end{aligned}$$

In case $x=X_{21}(a),y=X_{31}(b)$, if $z=t(c,d)$, then

$$\begin{aligned}J(x,y,z)) &= \psi(X_{21}(a),[X_{31}(b),t(c,d)])+\psi([X_{21}(a),t(c,d)],X_{31}(b))\\ &= \psi(X_{21}(a),X_{31}(b(cd-dc)))+\psi(X_{21}(a(cd-dc)),X_{31}(b))\\ &= (\overline{(ab(cd-dc)+a(cd-dc)b)})^{(-1)}=0.\end{aligned}$$

If $z=T_{12}(c,1)$, then

$$\begin{aligned}J(x,y,z)) &= \psi(X_{21}(a),[X_{31}(b),T_{12}(1,c)])+\psi([X_{21}(a),T_{12}(c,1)],X_{31}(b))\\ &= \psi(X_{21}(a),X_{31}(bc))+\psi(X_{21}(ca+ac),X_{31}(b))\\ &= (\overline{abc+(ca+ac)b})^{(-1)}=(\overline{3abc})^{(-1)}=0.\end{aligned}$$

If $z = T_{13}(c, 1)$, then

$$\begin{aligned} J(x, y, z)) &= \psi(X_{21}(a), [X_{31}(b), T_{13}(c, 1)]) + \psi([X_{21}(a), T_{13}(1, c)], X_{31}(b)) \\ &= \psi(X_{21}(a), X_{31}(cb + bc)) + \psi(X_{21}(ac), X_{31}(b)) \\ &= (\overline{a(cb + bc) + acb})^{(-1)} = (\overline{3abc})^{(-1)} = 0. \end{aligned}$$

As for the other subcases, they are similar to the above subcases except the following subcase: $x = X_{23}(a), y = X_{21}(b)$ and $z = t(c, d)$. In this situation, we have

$$\begin{aligned} J(x, y, z)) &= \psi(X_{23}(a), [X_{21}(b), t(c, d)]) + \psi([X_{23}(a), t(c, d)], X_{21}(b)) \\ &= \psi(X_{23}(a), X_{21}(b(cd - dc))) + \psi(0, X_{21}(b)) \\ &= -(\overline{ab(cd - dc)})^{(2)} = 0. \end{aligned}$$

Case 2: Suppose there is none of $\{x, y, z\}$ belonging to H. Then the nontrivial terms of $J(x, y, z)$ must be $\psi([X_{ik}(a), X_{kj}(b)], X_{ik}(c))$ or $\psi([X_{ik}(a), X_{kj}(b)], X_{kj}(c))$ for $a, b, c \in R$ and distinct i, j, k.

If: $x = X_{ik}(a)$, $y = X_{kj}(b)$ and $z = X_{ik}(c)$, then

$$\begin{aligned} J(x, y, z)) &= \psi(X_{ik}(a), [X_{kj}(b), X_{ik}(c)]) - \psi([X_{ik}(a), X_{kj}(b)], X_{ik}(c)) \\ &= \psi(X_{ik}(a), -X_{ij}(cb)) - \psi(X_{ij}(ab), X_{ik}(c)) \\ &= -\mathrm{sign}(k, j)(\overline{acb - abc})^{(i)} = 0. \end{aligned}$$

If $x = X_{ik}(a)$, $y = X_{kj}(b)$ and $z = X_{kj}(c)$, then

$$\begin{aligned} J(x, y, z)) &= \psi([X_{ik}(a), X_{kj}(c)], X_{kj}(b)) - \psi([X_{ik}(a), X_{kj}(b)], X_{kj}(c)) \\ &= \psi(X_{ij}(ac), X_{kj}(b)) - \psi(X_{ij}(ab), X_{kj}(c)) \\ &= \mathrm{sign}(i, k)(\overline{acb - abc})^{(-j)} = 0. \end{aligned}$$

The proof is completed. □

It is similar to the $\mathfrak{stl}_4(R)$ case, we obtain a central extension of $\mathfrak{stl}_3(R)$:

$$0 \to \mathcal{U} \to \widehat{\mathfrak{stl}}_3(R) \xrightarrow{\pi} \mathfrak{stl}_3(R) \to 0, \tag{4.2}$$

i.e.

$$\widehat{\mathfrak{stl}}_3(R) = \mathcal{U} \oplus \mathfrak{stl}_3(R), \tag{4.3}$$

and define $\mathfrak{stl}_3(R)^\sharp$ to be the Leibniz algebra generated by the symbols $X_{ij}^\sharp(a)$, $a \in R$ and

the K-linear space $\mathcal{U}$, satisfying the following relations:

$$X_{ij}{}^{\sharp}(k_1a+k_2b)=k_1X_{ij}{}^{\sharp}(a)+k_2X_{ij}{}^{\sharp}(b) \text{ for } a,b\in R,\ k_1,k_2\in K, \tag{4.4}$$
$$[X^{\sharp}_{ij}(a),X^{\sharp}_{jk}(b)]=-[X^{\sharp}_{jk}(b),X^{\sharp}_{ij}(a)]=X^{\sharp}_{ik}(ab) \text{ for distinct } i,j,k, \tag{4.5}$$
$$[X^{\sharp}_{ij}(a),\mathcal{U}]=0=[\mathcal{U},X^{\sharp}_{ij}(a)] \text{ for distinct } i,j, \tag{4.6}$$
$$[X^{\sharp}_{ij}(a),X^{\sharp}_{ij}(b)]=0 \text{ for distinct } i,j, \tag{4.7}$$
$$[X^{\sharp}_{ij}(a),X^{\sharp}_{ik}(b)]=\operatorname{sign}(j,k)(\overline{ab})^{i} \text{ for distinct } i,j,k, \tag{4.8}$$
$$[X^{\sharp}_{ij}(a),X^{\sharp}_{kj}(b)]=\operatorname{sign}(i,k)(\overline{ab})_{j} \text{ for distinct } i,j,k, \tag{4.9}$$

where $a,b\in R$, $1\le i,j,k\le 3$ are distinct. Then $\mathfrak{stl}_3(R)^{\sharp}$ is perfect and there is a unique Leibniz algebra homomorphism $\rho:\mathfrak{stl}_3(R)^{\sharp}\to\widehat{\mathfrak{stl}}_3(R)$.

As in Lemma 3.4, we have

Lemma 4.3 *$\rho:\mathfrak{stl}_3(R)^{\sharp}\to\widehat{\mathfrak{stl}}_3(R)$ is a Leibiz algebra isomorphism.*

Now we can state the main theorem of this section.

Theorem 4.4 *$(\widehat{\mathfrak{stl}}_3(R),\pi)$ is the universal central extension of $\mathfrak{stl}_3(R)$ and hence*

$$HL_2(\mathfrak{stl}_3(R))\cong\mathcal{U}.$$

Proof. The idea to prove this theorem is similar to that in the proof of Theorem 3.5. But there are some slight differences. The point is that since $1\le i,j,k\le 3$, if i,j,k are distinct, then k is uniquely determined once i,j are chosen.

Suppose

$$0\to\mathcal{V}\to\widetilde{\mathfrak{stl}}_3(R)\xrightarrow{\tau}\mathfrak{stl}_3(R)\to 0,$$

is a central extension of $\mathfrak{stl}_3(R)$. We must show that there exists a Leibniz algebra homomorphism $\eta:\widehat{\mathfrak{stl}}_3(R)\to\widetilde{\mathfrak{stl}}_3(R)$ so that $\tau\circ\eta=\pi$. Thus, by Lemma 4.3, it suffices to show that there exists a Leibniz algebra homomorphism $\xi:\mathfrak{stl}_3(R)^{\sharp}\to\widetilde{\mathfrak{stl}}_3(R)$ such that $\tau\circ\xi=\pi\circ\rho$. Choose a preimage $\widetilde{X}_{ij}(a)$ of $X_{ij}(a)$ as in Section 3, we need to check that relations (4.4)–(4.9) are satisfied.

Again set

$$\begin{aligned}&\widetilde{T}_{ij}(a,b)=[\widetilde{X}_{ij}(a),\widetilde{X}_{ji}(b)],\\&[\widetilde{X}_{ik}(a),\widetilde{X}_{kj}(b)]=\widetilde{X}_{ij}(ab)+\mu_{ij}(a,b),\\&[\widetilde{X}_{kj}(b),\widetilde{X}_{ik}(a)]=-\widetilde{X}_{ij}(ab)+\widetilde{\mu_{ji}}(b,a),\end{aligned}$$

for $a,b\in R$ and distinct i,j,k, where $\mu_{ij}(a,b),\widetilde{\mu_{ji}}(b,a)\in\mathcal{V}$. Then by Leibiz identity and the definition of $\widetilde{T}_{ij}(a,b)$, we have

$$[\widetilde{T}_{ij}(a,b),\widetilde{X}_{ik}(c)]=\widetilde{X}_{ik}(abc)+\mu_{ik}(a,bc).$$

On the other hand, we also have

$$\begin{aligned}
[\widetilde{T}_{ij}(a,b),\widetilde{X}_{ik}(c)] &= \Big[\widetilde{T}_{ij}(a,b),[\widetilde{X}_{ij}(1),\widetilde{X}_{jk}(c)]\Big] \\
&= \Big[[\widetilde{T}_{ij}(a,b),\widetilde{X}_{ij}(1)],\widetilde{X}_{jk}(c)\Big] - \Big[[\widetilde{T}_{ij}(a,b),\widetilde{X}_{jk}(c)],\widetilde{X}_{ij}(1)\Big] \\
&= [\widetilde{X}_{ij}(ab+ba),\widetilde{X}_{jk}(c)] - [-\widetilde{X}_{jk}(bac),\widetilde{X}_{ij}(1)] \\
&= \widetilde{X}_{ik}(abc) + \widetilde{X}_{ik}(bac) + \mu_{ik}(ab,c) + \mu_{ik}(ba,c) - \widetilde{X}_{ik}(bac) + \widetilde{\mu_{ki}}(bac,1) \\
&= \widetilde{X}_{ik}(abc) + \mu_{ik}(ab,c) + \mu_{ik}(ba,c) + \widetilde{\mu_{ki}}(bac,1).
\end{aligned}$$

So we get

$$\mu_{ik}(a,bc) = \mu_{ik}(ab,c) + \mu_{ik}(ba,c) + \widetilde{\mu_{ki}}(bac,1). \tag{4.10}$$

Taking $b=1$ in (4.10) gives

$$\mu_{ik}(a,c) + \widetilde{\mu_{ki}}(ac,1) = 0. \tag{4.11}$$

It follows that

$$\mu_{ik}(a,c) = -\widetilde{\mu_{ki}}(ac,1) = \mu_{ik}(1,ac) = \mu_{ik}(ac,1). \tag{4.12}$$

Similarly, by Leibniz identity, we have

$$\begin{aligned}
\Big[\widetilde{T}_{ij}(1,1),[\widetilde{X}_{ij}(a),\widetilde{X}_{jk}(b)]\Big] &= [\widetilde{T}_{ij}(1,1),\widetilde{X}_{ik}(ab)] = \widetilde{X}_{ik}(ab) + \mu_{ik}(1,ab) \\
&= \Big[[\widetilde{T}_{ij}(1,1),\widetilde{X}_{ij}(a)],\widetilde{X}_{jk}(b)\Big] - \Big[[\widetilde{T}_{ij}(1,1),\widetilde{X}_{jk}(b)],\widetilde{X}_{ij}(a)\Big] \\
&= \widetilde{X}_{ik}(ab) + 2\mu_{ik}(a,b) + \widetilde{\mu_{ki}}(b,a),
\end{aligned}$$

which yields

$$\mu_{ik}(1,ab) = 2\mu_{ik}(a,b) + \widetilde{\mu_{ki}}(b,a). \tag{4.13}$$

From (4.12) and (4.13), we have

$$\mu_{ik}(a,b) + \widetilde{\mu_{ki}}(b,a) = 0. \tag{4.14}$$

Now replacing $\widetilde{X}_{ij}(a)$ by $\widetilde{X}_{ij}(a) + \mu_{ij}(1,a)$ which satisfies (4.4), we have at once

$$[\widetilde{X}_{ik}(a),\widetilde{X}_{kj}(b)] = \widetilde{X}_{ij}(ab), \tag{4.15}$$
$$[\widetilde{X}_{kj}(b),\widetilde{X}_{ik}(a)] = -\widetilde{X}_{ij}(ba), \tag{4.16}$$

for $a,b \in R$, distinct i,j,k. This gives (4.5). The proof of relation (4.7) is exactly the same as (4.13).

To show $\widetilde{X}_{ij}(a)$ satisfies (4.8) and (4.9), we define

$$[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] = \nu^{i}_{jk}(a,b) \quad \text{and} \quad [\widetilde{X}_{ij}(a), \widetilde{X}_{kj}(b)] = \nu^{ik}_{j}(a,b),$$

where $\nu^{i}_{jk}(a,b), \nu^{ik}_{j}(a,b) \in \mathcal{V}$. Then

$$\begin{aligned}\nu^{i}_{jk}(a,b) &= [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] = \Big[\widetilde{X}_{ij}(a), [\widetilde{X}_{ij}(1), \widetilde{X}_{jk}(b)]\Big] \\ &= -[\widetilde{X}_{ik}(ab), \widetilde{X}_{ij}(1)] = -\nu^{i}_{kj}(ab,1).\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\nu^{i}_{jk}(a,b) &= [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(b)] = \Big[[\widetilde{X}_{ik}(a), \widetilde{X}_{kj}(1)], \widetilde{X}_{ik}(b)\Big] \\ &= [\widetilde{X}_{ik}(a), -\widetilde{X}_{ij}(b)] = -\nu^{i}_{kj}(a,b) \\ &= \Big[[\widetilde{X}_{ik}(1), \widetilde{X}_{kj}(a)], \widetilde{X}_{ik}(b)\Big] \\ &= [\widetilde{X}_{ik}(1), -\widetilde{X}_{ij}(ba)] = -\nu^{i}_{kj}(1,ba).\end{aligned} \tag{4.18}$$

So we have

$$\nu^{i}_{jk}(a,b) = -\nu^{i}_{kj}(a,b) = -\nu^{i}_{kj}(ab,1) = -\nu^{i}_{kj}(1,ba). \tag{4.19}$$

Moreover, we have

$$\begin{aligned}0 &= \Big[\widetilde{T}_{ij}(1,1), [\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(1)]\Big] \\ &= \Big[[\widetilde{T}_{ij}(1,1), \widetilde{X}_{ij}(a)], \widetilde{X}_{ik}(1)\Big] - \Big[[\widetilde{T}_{ij}(1,1), \widetilde{X}_{ik}(1)], \widetilde{X}_{ij}(a)\Big] \\ &= 2[\widetilde{X}_{ij}(a), \widetilde{X}_{ik}(1)] - [\widetilde{X}_{ik}(1), \widetilde{X}_{ij}(a)] \\ &= 2\nu^{i}_{jk}(a,1) - \nu^{i}_{kj}(1,a) \\ &= 2\nu^{i}_{jk}(1,a) + \nu^{i}_{jk}(1,a) = \nu^{i}_{jk}(1,3a).\end{aligned} \tag{4.20}$$

Similarly, we have

$$\nu^{ik}_{j}(a,b) = \nu^{ik}_{j}(b,a) = \nu^{ik}_{j}(ab,1) = -\nu^{ki}_{j}(a,b) = -\nu^{ki}_{j}(ba,1), \tag{4.21}$$

and

$$\nu^{ik}_{j}(1,3a) = 0. \tag{4.22}$$

Set $\widetilde{t}(a,b) = \widetilde{T}_{1j}(a,b) - \widetilde{T}_{1j}(ba,1)$, which does not depend on the choice of j. By the Leibniz identity and (4.19)–(4.22), we have

$$\begin{aligned}0 &= \Big[\widetilde{t}(a,b), [\widetilde{X}_{12}(1), \widetilde{X}_{13}(c)]\Big] \\ &= \Big[[\widetilde{t}(a,b), \widetilde{X}_{12}(1)], \widetilde{X}_{13}(c)\Big] - \Big[[\widetilde{t}(a,b), \widetilde{X}_{13}(c)], \widetilde{X}_{12}(1)\Big] \\ &= [\widetilde{X}_{12}(ab-ba), \widetilde{X}_{13}(c)] - [\widetilde{X}_{13}((ab-ba)c), \widetilde{X}_{12}(1)] \\ &= \nu^{1}_{23}(ab-ba,c) - \nu^{1}_{32}((ab-ba)c,1) = \nu^{1}_{23}((ab-ba)c,1) + \nu^{1}_{23}((ab-ba)c,1) \\ &= \nu^{1}_{23}(2(ab-ba)c,1) = \nu^{1}_{23}(1,2(ab-ba)c) = -\nu^{1}_{23}(1,(ab-ba)c),\end{aligned}$$

i.e. $\nu^1_{23}(1,(ab-ba)c)=0$. We also have

$$\begin{aligned}0&=\Big[\widetilde{t}(a,b),[\widetilde{X}_{12}(1),\widetilde{X}_{32}(c)]\Big]\\&=\Big[[\widetilde{t}(a,b),\widetilde{X}_{12}(1)],\widetilde{X}_{32}(c)]\Big]\\&=[\widetilde{X}_{12}(ab-ba),\widetilde{X}_{32}(c)]\\&=\nu^{13}_2(ab-ba,c)=\nu^{13}_2(c,ab-ba)=\nu^{13}_2((ab-ba)c,1)=\nu^{13}_2(1,(ab-ba)c).\end{aligned}$$

More generally,

$$\nu^i_{jk}(1,(ab-ba)c)=0 \text{ and } \nu^{ik}_j(1,(ab-ba)c)=0, \tag{4.23}$$

for $a,b,c\in R$ and distinct $1\le i,j,k\le 3$. Above discussions prove

$$\nu^i_{jk}(1,\mathcal{I}_3)=0 \text{ and } \nu^{ik}_j(1,\mathcal{I}_3)=0. \tag{4.24}$$

Now similar arguments as in [GS] complete the proof of the theorem. □

References

[AF] B. N. Allison and J. R. Faulkner, *Nonassociative coefficient algebras for Steinberg unitary Lie algebras,* J. Algebra 161 (1993) 1–19.

[Bl] S. Bloch, *The dilogarithm and extensions of Lie algebras,* Alg. K-theory, Evanston 1980, Springer Lecture Notes in Math 854 (1981) 1–23.

[G1] Y. Gao, *Steiberg Unitary Lie Algebras and Skew-Dihedral Homology,* J.Algebra,17 (1996),261–304.

[G2] Y. Gao, *On the Steinberg Lie algebras $st_2(R)$,* Comm. Alg. 21 (1993) 3691–3706.

[GS] Y. Gao and S. Shang, *Universal coverings of Steinberg Lie algebras of small characteristic,* preprint, math.QA/0512188.

[Ka] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative ring,* J. Pure and Appl. Alg. 34 (1984) 265–275.

[KL] C. Kassel and J-L. Loday, *Extensions centrales d'algèbres de Lie,* Ann. Inst. Fourier 32 (4) (1982) 119–142.

[L] Liu, Dong, *Steinberg Leibniz algebras and superalgebras,* J. Alg., Vol. **283**(1) (2005), 199-221.

[Lo] J-L. Loday, *Cyclic homology,* Grundlehren der mathematischen Wissenschaften 301, Springer 1992.

[LP] J-L. Loday and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology,* Math. Ann. 296 (1993), 139-158.